Initial and final de Sitter universes from modified $f(R)$ gravity

G. Cognola[1], E. Elizalde[2], S.D. Odintsov[3], P. Tretyakov[4] and S. Zerbini[1]

[1] *Dipartimento di Fisica, Università di Trento and Istituto Nazionale di Fisica Nucleare Gruppo Collegato di Trento, Italia*

[2] *Consejo Superior de Investigaciones Científicas ICE/CSIC-IEEC, Campus UAB, Facultat de Ciències, Torre C5-Parell-2a pl, E-08193 Bellaterra (Barcelona) Spain*

[3] *Institució Catalana de Recerca i Estudis Avançats (ICREA) and Institut de Ciencies de l'Espai (IEEC-CSIC), Campus UAB, Facultat de Ciències, Torre C5-Parell-2a pl, E-08193 Bellaterra (Barcelona) Spain and*

[4] *JINR, Dubna, Moscow region, Russia*

(Dated: January 15, 2009)

Viable models of modified gravity which satisfy both local as well as cosmological tests are investigated. It is demonstrated that some versions of such highly non-linear models exhibit multiply de Sitter universe solutions, which often appear in pairs, being one of them stable and the other unstable. It is explicitly shown that, for some values of the parameters, it is possible to find several de Sitter spaces (as a rule, numerically); one of them may serve for the inflationary stage, while the other can be used for the description of the dark energy epoch. The numerical evolution of the effective equation of state parameter is also presented, showing that these models can be considered as natural candidates for the unification of early-time inflation with late-time acceleration through dS critical points. Moreover, based on the de Sitter solutions, multiply SdS universes are constructed which might also appear at the (pre-)inflationary stage. Their thermodynamics are studied and free energies are compared.

PACS numbers: 11.25.-w, 95.36.+x, 98.80.-k

I. INTRODUCTION

Modified gravity (for a review, see e.g. [1]) constitutes an interesting dynamical alternative to ΛCDM cosmology in that it is also able to describe with success the current acceleration in the expansion of our Universe, the present dark energy epoch. The specific class of modified $f(R)$ gravities (for a review, see e.g. [1, 2]) has undergone many studies which suggest that this family of gravitational alternatives for dark energy is able to pass the constringent solar system tests. The investigation of cosmic acceleration as well as the study of the cosmological properties of $f(R)$ models has been done in Refs. [1, 2, 3, 4, 5, 6]. The possibility of a natural unification of early-time inflation with late-time acceleration becomes a realistic and quite natural possibility in such models, as is demonstrated e.g. in Ref. [3].

Recently, the importance of modified gravity models of this kind has been reassessed with the appearance of the so-called 'viable' $f(R)$ models [7, 8, 9, 10, 11]. Those are theories which satisfy both the cosmological as well as the local gravity constraints, which had caused in the past a number of serious problems to some of the first-generation theories, that had to be considered now as only approximate descriptions from more realistic theories. The final aim of all those phenomenological models is to describe a segment as large as possible of the entire history of our universe, as well as to recover all local predictions of Einstein's gravity that have been already verified experimentally, with very good accuracy, at the solar system scale. It is remarkable that, as was demonstrated in Refs. [9, 10], some of these realistic models lead to a natural unification of the early-time inflation epoch with the late-time acceleration stage.

Let us recall that, in general (see e.g. [1, 2], for a review), the total action for the modified $f(R)$ gravitational models can be written as

$$S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g}\,[R - F(R)] + S_{(m)}\,. \tag{1}$$

Here $F(R)$ is a suitable function of the scalar curvature R, which defines the modified gravitational part of the model. The general equation of motion in $f(R) \equiv R - F(R)$ gravity with matter is given by

$$\frac{1}{2} g_{\mu\nu} f(R) - R_{\mu\nu} f'(R) - g_{\mu\nu} \Box f'(R) + \nabla_\mu \nabla_\nu f'(R) = -\frac{\kappa^2}{2} T_{(m)\mu\nu}\ , \tag{2}$$

where $T_{(m)\mu\nu}$ is the matter energy-momentum tensor and $f'(R)$ is the derivative of $f(R)$ with respect to its argument R. For a generic $f(R)$ model is not easy to find exact static solutions. However, if one impose some restrictions, one can proceed along the following lines. First, we may require the existence of solutions with *constant* scalar curvature $R = R_0$, and we arrive at

$$f'(R_0)R_{\mu\nu} = \frac{f(R_0)}{2} g_{\mu\nu} \,. \tag{3}$$

Taking the trace, we have the condition

$$2f(R_0) = R_0\, f'(R_0) \tag{4}$$

and this means that the solutions are Einstein's spaces, namely they have to satisfy the equation

$$R_{\mu\nu} = \frac{f(R_0)}{2f'(R_0)} g_{\mu\nu} = \frac{R_0}{4} g_{\mu\nu} \,, \tag{5}$$

R_0 being a solution of Eq. (4). This gives rise to an effective cosmological constant, namely

$$\Lambda_{eff} = \frac{f(R_0)}{2f'(R_0)} = \frac{R_0}{4} \,. \tag{6}$$

The purpose of our work will be to study the appearance of multiply de Sitter space solutions in several realistic models of modified gravity. The occurrence of multiply de Sitter solutions plays a fundamental role in modified gravity because it permits to describe the inflation stage as well as current ΛCDM cosmology in terms of this theory alone without any need for either fine-tuning of a cosmological constant nor of introducing extra scalar fields. In summary, this is a minimal and at the same time very powerful approach which circumvents some of the hardest problems of present day physics. One may argue that these theories are equivalent to introducing extra scalar fields, but this equivalence has been proven to hold at the classical level only, not at the quantum one. In addition, the cosmological interpretation of modified gravity solutions is different from that of scalar field cosmology.

The paper is organized as follows. In the next section we discuss a viable modified gravity model [7] in an attempt to study de Sitter solutions there. It is shown that, for some values of the parameters, it is possible to find several de Sitter spaces (as a rule, numerically, for small values of the curvature exponent); one of them may serve for the inflationary stage, while the other one can be used for the description of the dark

energy stage. The evolution of the effective equation of state parameter is investigated numerically. Sect. 3 is devoted to the analysis of the same question in a slightly generalized model which is known to describe the unification of the early-time inflation with the late-time acceleration epochs [9]. The same numerical investigation is carried out, and a multiply de Sitter universe solution is constructed. In section 4, the corresponding problem is investigated for a viable model of tangential modified gravity, proposed in Ref. [11]. Having in mind the possibility to use a SdS universe for the description of cosmic acceleration, those solutions for modified gravity are investigated in Sect. 5. Comparison of the free energies of the de Sitter and the SdS solutions, for the viable model of the second section, is then made. Finally, some outlook is given, together with the conclusions, at the end of the paper.

II. DYNAMICAL SYSTEM APPROACH AND DE SITTER SOLUTIONS IN REALISTIC MODIFIED GRAVITY

In this section we will study de Sitter solutions in realistic modified gravity models using a dynamical system approach. We shall start from the following form for the initial action

$$S = \int d^4x\sqrt{-\mathrm{g}}\left[\frac{1}{2\kappa^2}R - F(A)\right], \tag{7}$$

where A is some function of the geometrical invariants. In order to make use of the method of dynamical systems, we work in the metric corresponding to a spatially-flat FRW universe, namely

$$\mathrm{g}_{\mu\nu} = diag(-n(t)^2, a(t)^2, a(t)^2, a(t)^2). \tag{8}$$

From here, the FRW equation can be written in the following way (for more details see Ref. [12])

$$\frac{3}{\kappa^2}H^2 = \rho_F, \tag{9}$$

where

$$\rho_F = F + F'A_n - 3HF'A_{\dot{n}} - F'\frac{dA_{\dot{n}}}{dt} - F''A_{\dot{n}}\dot{A}. \tag{10}$$

The "dot" over the symbol means derivative with respect to time t, while A_n and $A_{\dot{n}}$ represent the derivative of A with respect to n and $\dot{n}$ respectively. In what follows we will

mainly concentrate on the simplest case $A = R$.

Let us consider the following choice for the function F [7], which represents a very interesting subclass of viable modified gravitational models

$$F(R) = \frac{\mu^2}{2\kappa^2} \frac{c_1(\frac{R}{\mu^2})^k + c_3}{c_2(\frac{R}{\mu^2})^k + 1}, \tag{11}$$

where the constant μ has dimension of mass, while c_1, c_2, c_3 are some positive dimensionless constants. Note that here $R = \frac{6}{n^2}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}}{a}\frac{\dot{n}}{n} + \frac{\dot{a}^2}{a^2}\right]$, $R_n = -2R = -12(\dot{H} + 2H^2)$, $R_{\dot{n}} = -6H$. After making variation, it is chosen $n = 1$. Introducing a nonzero constant c_3 is here equivalent to introducing a shift in the effective cosmological constant, and for this reason we will assume $c_3 = 0$ in our further computations. Finally, we can write Eq. (9) for the function in (11) as

$$\begin{aligned} \frac{6}{\mu^2}H^2 = & \frac{c_1\left(\frac{R}{\mu^2}\right)^k}{\left(c_2\left(\frac{R}{\mu^2}\right)^k+1\right)} - \frac{6(\dot{H}+H^2)\frac{kc_1}{\mu^2}\left(\frac{R}{\mu^2}\right)^{k-1}}{\left(c_2\left(\frac{R}{\mu^2}\right)^k+1\right)^2} + \\ & \frac{36(H\ddot{H}+4H^2\dot{H})}{\left(c_2\left(\frac{R}{\mu^2}\right)^k+1\right)^3}\frac{kc_1}{\mu^4}\left[(k-1)\left(\frac{R}{\mu^2}\right)^{k-2} - c_2(k+1)\left(\frac{R}{\mu^2}\right)^{2k-2}\right], \end{aligned} \tag{12}$$

where $R = 6(\dot{H} + 2H^2)$. This equation can be rewritten as a dynamical system, namely

$$\begin{aligned} \dot{H} &= C, \\ \dot{C} &= F_1(H, C). \end{aligned} \tag{13}$$

It is easy to see that the critical points of this system are the de Sitter points ($\dot{H} = 0$, $\ddot{H} = 0$). To investigate the nature of these points we need to determine them explicitly, a non-trivial problem in the general case. It is however easy to obtain from (12) the equation satisfied by the critical points H_0:

$$\frac{6}{\mu^2}H_0^2 = \frac{c_1\left(\frac{12H_0^2}{\mu^2}\right)^k}{\left(c_2\left(\frac{12H_0^2}{\mu^2}\right)^k + 1\right)} - \frac{6H_0^2\frac{kc_1}{\mu^2}\left(\frac{12H_0^2}{\mu^2}\right)^{k-1}}{\left(c_2\left(\frac{12H_0^2}{\mu^2}\right)^k + 1\right)^2}. \tag{14}$$

The same result can be obtained directly starting from equation (4) of the previous Section.

It is convenient to introduce the notation $x_0 \equiv 12\frac{H_0^2}{\mu^2}$, for further investigation of this equation. So, finally, we have

$$c_2^2 x_0^{2k+1} - 2c_1c_2x_0^{2k} + 2c_2x_0^{k+1} + 2c_1(\frac{1}{2}k - 1)x_0^k + x_0 = 0. \tag{15}$$

First of all we find one (trivial) root of this equation $x_0 = 0$ (it corresponds to $H_0 = 0$), what allows us to reduce the order of the equation. (Note also that, if $c_3 \neq 0$, Eq. (15) takes a more complicate form and then it does not have the trivial root $x_0 = 0$.) But nevertheless the equation still is of $2k$-order and in the interesting case ($k > 2$); this is too high and the roots cannot be found algebraically. We need a specific discussion of this problem. It is clear that if there are too many (say 10 or 20) de Sitter points in the theory, it looks like a classical analogue (at a reduced scale, of course) [13] of the string landscape vacuum structure in which case it will be far from trivial to obtain the standard cosmology. Nevertheless, something can be done even in this case, by comparing the energies of the corresponding de Sitter solutions, which should in fact differ, as is discussed in Sect. V. Using Descartes rule of signs, we find that Eq. (15) can have 2 or 0 positive roots. More detailed information can be obtained by using Sturm's theorem. Unfortunately it is not possible to apply it in the general case (for arbitrary k), so we need to investigate separately each of the different cases, $3 \leqslant k \leqslant 10$, and see which one is more interesting from a cosmological viewpoint. We have therefore undertaken here a systematic analysis of all possible cases in this region of values of k.

Computations are rather involved (specially for larger k), but the final results are not so difficult to describe analytically. We have found, for any of the values of k considered, in the range above, that the number of roots depends on the parameter $\alpha_k = \frac{c_1^k}{c_2^{k-1}}$ only, and that there is an α_k^* such that, for $0 < \alpha_k < \alpha_k^*$, Eq. (15) has no positive roots, while for $\alpha_k > \alpha_k^*$, Eq. (15) has two positive roots. We here now enumerate our results systematically: $\alpha_3^* = -\frac{3^3 11}{2} + \frac{3^3 5}{2}\sqrt{5} \approx 2.43$; $\alpha_4^* = \frac{1237}{2^5} - \frac{1837}{2^5 3^2}\sqrt{33} \approx 2.01$; $\alpha_5^* = -\frac{5^5}{2^2 3^4} + \frac{5^5}{3^5 2}\sqrt{3} \approx 1.49$; $\alpha_6^* = \frac{3^7 17\cdot 6977}{2^{24} 5} - \frac{2\cdot 8329\cdot 32183}{2^{24} 5^3}\sqrt{65} \approx 1.03$; $\alpha_7^* = -\frac{7^7 17681}{2\cdot 3^3 5^{13}} + \frac{7^9 107}{2\cdot 3^2 5^{13}}\sqrt{21} \approx 0.68$; $\alpha_8^* = \frac{69401}{2\cdot 3^7 7^2} + \frac{5\cdot 66821}{2\cdot 3^8 7^4}\sqrt{105} \approx 0.43$; $\alpha_9^* = \frac{3^{18}\cdot 719\cdot 24709}{2^8 7^{17}} + \frac{3^{18} 517399}{2^3 7^{17}}\sqrt{2} \approx 0.26$; $\alpha_{10}^* = \frac{5^{10} 401843202307}{2^{59} 3^4} + \frac{5^{10} 17\cdot 67\cdot 179\cdot 659\cdot 164429}{2^{59} 3^9}\sqrt{17} \approx 0.16$. Note also that, near the critical point α_k^*, both solution are very close to each other when they exist (a real value of $\alpha_k > \alpha_k^*$), and they are complex conjugate to each other for $0 < \alpha_k < \alpha_k^*$. This means that, in a situation where these two roots differ substantially, it must necessarily be $\alpha_k \gg \alpha_k^*$.

The dS-points described above can be used for the construction of inflationary or late-time acceleration behavior (depending on the value of the scale factor μ^2). To this aim, the corresponding dS point must be unstable in the inflationary stage but can be either

unstable or stable for the late-time acceleration one. However, this is in fact not a strictly necessary condition, since even for stable dS inflation, the exit from it can be achieved by a coupling with matter, through the effect of small non-local term or by some other mechanism. Unfortunately, an all-round investigation of stability of the dS point turns out to be very difficult in the general case. For the recent analysis of critical points in more general modified gravity theories depending on all geometrical invariant see Ref. [14].

Here we can carry out the analysis on the stability of de Sitter points for the model we are dealing with in this section, only for some chosen numerical set of parameters. Such computations show that one of the dS points is very likely to be always unstable (with a smaller value of H) and that another one is very likely to be always stable (with larger value of H). This means that there is actually an easy way for the inflationary stage construction (if the initial conditions lie sufficiently close to the unstable dS point, see Fig. 1). However, only within the model under consideration it might be a problem with late-time acceleration, originated from the stable point. The only possibility for late time acceleration is to choose very special parameters, in which the stable point is situated very far away from the unstable one. Of course, the problem disappears when one describes only late-time acceleration within such a model, or when one takes into account other terms like the local ones. Note however that specific values of the parameters in which both points are unstable can be chosen too, which give rise to a sound theory which provides a unified description of the inflationary and late-time acceleration epochs.

Let us now consider specific numerical results. First, note that a full numerical investigation starting from the inflation (or from our dS point) and extending to the late-time acceleration epoch and going through a FRW-like stage is not feasible because of the presence of numerical instabilities that prevent such possibility. For this reason, we will investigate the regions near the equilibrium points only, and especially near the unstable one. In Fig. 1 the trajectories in the phase plane $(H - \dot{H})$ are presented for the set of parameters: $\mu = 1$, $k = 4$, $c_1 = 1.12 \cdot 10^{-3}$, $c_2 = 6.21 \cdot 10^{-5}$. Such small values of c_1 and c_2 are motivated in [7], but the picture is typical in any case. The position of the dS point is specified. We may see one stable point (with bigger value of H) and another, unstable one. The stars in the graph correspond to the beginning of the trajectories. We see clearly that trajectories can pass sufficiently close to the dS point, and that they escape from

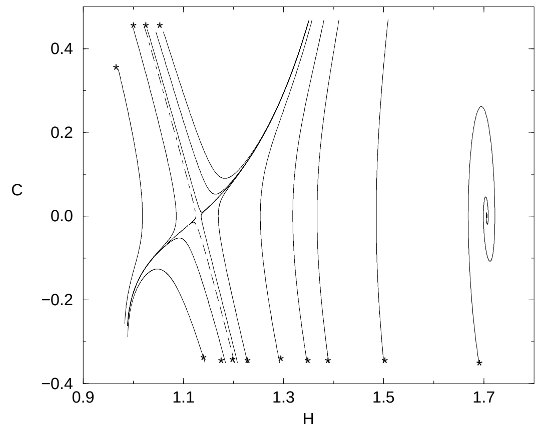

FIG. 1: Phase plane ($H - C \equiv \dot{H}$) near the equilibrium points. Stars denote the beginning of the trajectories. Dashed and dot dashed lines lead to bottom.

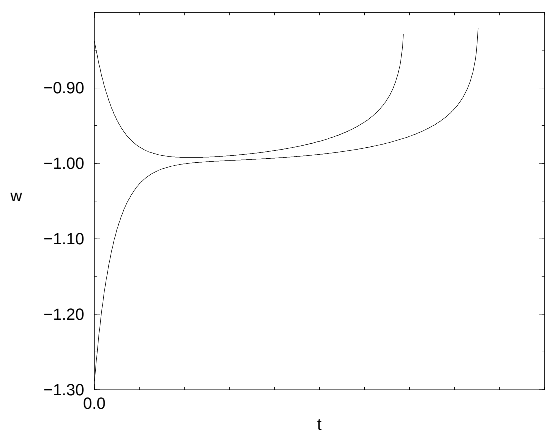

FIG. 2: Evolution of w_{eff} near an unstable point for the dashed and dot-dashed lines of Fig. 1. The time origin corresponds to the stars from the previous graph.

it, lead by the two attractors. One of them (the upper one) leads to a singularity, while another one (heading towards the bottom), leads to a solution going somewhere in the FRW-like region (this is difficult to analyze exactly, due to numerical instabilities). In Fig. 2 the evolution of $w_{eff} \equiv -1 - \frac{2\dot{H}}{3H^2}$ is depicted for two trajectories: the upper one corresponds to the dashed line of Fig. 1, the bottom one to the dot-dashed line of Fig. 1 (both of them lead to the bottom in Fig. 1). We see that there actually are trajectories

which start from phantom-like solutions and tend to a normal evolution (when this is possible in the model). Of course, these are qualitative considerations only, owing to the fact that no matter is taken into account. Most probably, inclusion of matter will change the situation. However, it is just remarkable that both the early-time inflation epoch as well as the late-time accelerating one could be obtained in a unified and natural way in such a model, owing to the presence of several dS points.

III. A DIFFERENT EXAMPLE OF $f(R)$ VIABLE MODEL

A slightly different choice for the function $F(R)$ is motivated by the realistic and viable model which was proposed for the unification of early-time inflation and late-time acceleration in Ref. [9], namely

$$F(R) = \frac{(R-R_0)^m + R_0^m}{f_0 + f_1\left[(R-R_0)^m + R_0^m\right]}. \tag{16}$$

Here we imply that f_0, f_1 and R_0 is positive, what seems reasonable. Now, our FRW-like equation (9) reads

$$\begin{aligned}&\frac{3}{\kappa^2}H^2 = \frac{\{(R-R_0)^m+R_0^m\}}{\left(f_0+f_1\{(R-R_0)^m+R_0^m\}\right)} - \frac{6(\dot{H}+H^2)f_0 m(R-R_0)^{m-1}}{\left(f_0+f_1\{(R-R_0)^m+R_0^m\}\right)^2}+\\ &\frac{36(H\ddot{H}+4H^2\dot{H})f_0 m(R-R_0)^{m-2}}{\left(f_0+f_1\{(R-R_0)^m+R_0^m\}\right)^3}\left[(m-1)(f_0+f_1R_0^m)-(m+1)f_1(R-R_0)^m\right],\end{aligned} \tag{17}$$

where $R = 6(\dot{H}+2H^2)$. Rewriting this equation under the form of a dynamical system (13) we find, by introducing the new variable $x \equiv 12H_0^2 - R_0$, that the equilibrium points of this system (which are dS points indeed) are obtained from the following equation

$$\begin{aligned}&f_1^2x^{2m+1} + f_1(f_1R_0 - 4\kappa^2)x^{2m} + 2f_1(f_0+f_1R_0^m)x^{m+1}+\\ &\left[2R_0f_1(f_0+f_1R_0^m) - 8\kappa^2 f_1R_0^m + 2\kappa^2(m-2)f_0\right]x^m + 2\kappa^2R_0f_0mx^{m-1}+\\ &(f_0+f_1R_0^m)^2x + R_0(f_0+f_1R_0^m)(f_0+f_1R_0^m - 4\kappa^2R_0^{m-1}) = 0.\end{aligned} \tag{18}$$

It is clear that, in the general case (even for fixed $m>2$), solving this equation will not be an easy thing. We can make use of the following hint. Let us require that $x=0$ be a solution of Eq. (18). This is possible only if there is some relation among the constants

of our theory. As we can easily see from (18), it must be that $f_0 = R_0^{m-1}(4\kappa^2 - f_1 R_0)$. Moreover, since $f_0 > 0$, we need that $f_1 R_0 < 4\kappa^2$. Substituting this condition into (18) we find an essential simplification of the former equation, namely

$$f_1^2 x^{2m} - f_1(4\kappa^2 - f_1 R_0)x^{2m-1} + 8\kappa^2 f_1 R_0^{m-1} x^m +$$

$$2\kappa^2(m-2)R_0^{m-1}(4\kappa^2 - f_1 R_0)x^{m-1} + 2\kappa^2 R_0^m(4\kappa^2 - f_1 R_0)mx^{m-2} + (4\kappa^2 R_0^{m-1})^2 = 0, \tag{19}$$

where the root $x = 0$ is already excluded. We now discuss the physical meaning of finding a solution $x = 0$. As a rule this will imply that R_0 is the value of the scalar curvature at present time, but in principle R_0 is just a parameter of the theory, which could be given any likely value. We can, for instance, fix R_0 to have the dS point value which is situated exactly at present time, t_0, or at $t_0 + 100$ years. And there is also the possibility to explain late time acceleration if this point is stable. Now let us consider other possible roots of Eq. (19). Using Sturm's theorem as in the previous section, we find several positive roots (which correspond to dS points in the past) and a number of negative roots (needless to say, only values which are $x > -R_0$ have a physical meaning), corresponding to dS points in the future. Unfortunately, computations are harder than in the previous case and our calculations have been performed for three specific values of the parameter m: 3, 5 and 7, only. In any case, the results are very similar for the three situations, what gives us hope that for larger values of m result will be also similar. We may predict the upper limit of the dS points in the future only, which is 2, but in any case this point is not much interesting.

Concerning the dS points in the past, the picture turns out to be much like the one in the previous case: the number of dS points depends only on a dimensionless parameter for all investigated values of m. That is, $\beta = \frac{2\kappa^2}{f_1 R_0}$. And there are two values, β_* and β_{**}, which are different for each m, so for $0 < \beta < \beta_*$ and $\beta > \beta_{**}$ there are two dS points and for $\beta_* < \beta < \beta_{**}$ there is no dS point. We have not been able to obtain exact analytical expressions for β_* and β_{**} and thus give here numerical results only. For $m = 3$ $\beta_* \approx 0.153$, $\beta_{**} \approx 3.12$ (as positive roots of $4\beta^4 + 336\beta^3 - 843\beta^2 - 815\beta + 144 = 0$). For $m = 5$ $\beta_* \approx 0.163$, $\beta_{**} \approx 2.01$ (as positive roots of $1318032\beta^8 + 40509072\beta^7 - 208593144\beta^6 + 472402800\beta^5 - 637548615\beta^4 + 422702939\beta^3 - 90089631\beta^2 - 69190983\beta + 12301875 = 0$). For

$m = 7$ $\beta_* \approx 0.215$, $\beta_{**} \approx 1.68$ (as positive roots of $2^6 5^9 24337\beta^{11} + 2^6 5 \cdot 181772596417\beta^{10} - 2^4 7 \cdot 17 \cdot 29 \cdot 641 \cdot 15074567\beta^9 + 2^5 19 \cdot 29 \cdot 173 \cdot 6421 \cdot 100207\beta^8 - 2^2 7 \cdot 153140941633579\beta^7 + 2^2 7 \cdot 227057313819467\beta^6 - 5 \cdot 7 \cdot 195531002131489\beta^5 + 2^2 47 \cdot 1331611 \cdot 21800759\beta^4 - 2 \cdot 5 \cdot 53 \cdot 2797 \cdot 2102824303\beta^3 + 2^3 7 \cdot 29 \cdot 728096696819\beta^2 - 5^4 23 \cdot 18382295597\beta + 2^{16} 5^5 7^6 = 0$).

As we can see, the interval where there are no dS points shrinks when the value of the parameter m is increased. If we assume that for larger m the number of dS points depends on the parameter β only—as clearly happens in the investigated cases—we do find β_* and β_{**} for any value of m. Such investigations show that β_* and β_{**} slowly change when m increases and that the interval without dS points is present even for very big value of the parameter m. For example, for $m = 101$ we have $\beta_* \approx 0.480$ and $\beta_{**} \approx 1.06$. A simple numerical investigation of stability for the existing dS points shows that one of them (the one with the smaller value of H) is always unstable, but the other one (with the larger value of H) can either be stable or unstable. From our numerical results, the appearance of unstable dS points pairs relative to the interval $0 < \beta < \beta_*$ seems most probable. Note also that the condition $f_0 > 0$ means that it actually must be $\beta > 0.5$ and that in this case it is most likely the one stable and one unstable point show up.

Now let us consider the evolution equation (17) from a different point of view. As we already noted, R_0 is a parameter of the theory which corresponds to the value of the scalar curvature at some epoch. This means that during the normal evolution of our universe, from large to little (or zero) curvature, it becomes $R = R_0$ at some moment. But, as we can see from Eq. (17), this means that that at this point the coefficient of the higher derivative term ($\ddot{H}$) is equal to zero. This is a well-known mathematical problem, which needs special investigation. As we know from mathematics there are two possibilities: the solution of the perturbed equation (with a higher derivative term) may tend to the solution of the degenerate equation (without higher derivatives), and then the coefficient of the higher derivative term may tend to zero or not tend to the solution of the degenerate equation. A special investigation of this problem shows that solutions of Eq. (17) tend to solutions of the degenerate equation, which is

$$\frac{3}{\kappa^2} H_d^2 = \frac{R_0^m}{(f_0 + f_1 R_0^m)}, \tag{20}$$

when $R = R_0 + 0$ but not so when $R = R_0 - 0$. This means that the point R_0 is reachable during the evolution from large R to zero, but it is not the final point of the evolution,

because there is an instability in the future. On the other hand, we have $R = R_0$ at this point, and therefore $3H_d^2 = R_0/4$. Substituting this into (20) we find a relation among the parameters of the theory which were introduced before, $f_0 = R_0^{m-1}(4\kappa^2 - f_1 R_0)$. This means that, generically, if one wants the point R_0 to be reachable during evolution, a relation of this sort among the parameters of the theory must be fulfilled. That is, we have two independent parameters only (or even just one if we consider that R_0 is strictly related to our epoch). Note also that it is impossible to use standard numerical methods near this point for solving Eq. (17) because all available methods need solving the equation with respect to the highest derivative, and it turns out that, near this point, the numerical solution is unstable.

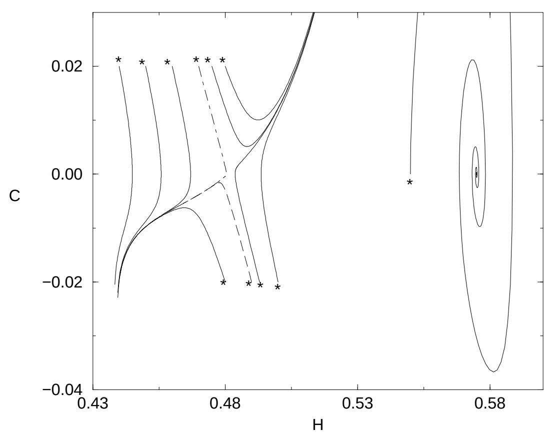

FIG. 3: Phase plane ($H - C \equiv \dot{H}$) near equilibrium points. Stars denote beginning of trajectories. Dashed and dot dashed lines lead to bottom.

Now let us consider numerical results near the stable points. Since the value of R_0 is very small in our epoch ($\sim 10^{-56}$ cm^{-2}), it is impossible to use real values in computations. Thus, we have produced a qualitative analysis using the following set of parameters: $R_0 = 1$, $f_1 = 1$, $\kappa^2 = 1$, $m = 7$. As we can see, $\beta = 2$, and there must be two dS points corresponding to positive x. The evolution lines near these points are represented in Fig. 3, where we can trace one stable and one unstable point, stars denoting the beginning of trajectories, as in the previous case. Actually, there are three dS points for these chosen parameters. One of them, $H = 0.288$, which corresponds to $x = 0$, is degenerate and situated out of the graph. The time evolution of w_{eff} is represented in

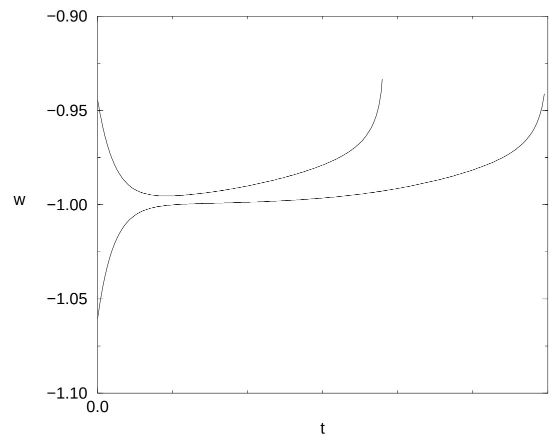

FIG. 4: Evolution of w_{eff} near unstable point for dashed and dot-dashed lines of Fig.1. Zero in time corresponds to the stars from the previous graph.

Fig. 4 by the dashed and dot-dashed lines. The picture is very similar to the one in the previous section, and the same comments also apply here.

Thus, we have here shown that the unification of early-time inflation with late-time acceleration is in principle possible, and even very likely, due to the appearance of several dS points in the evolution of the universe. For a more realistic study, the presence of (minimally and non-minimally coupled) matter should be taken into account with care.

IV. DE SITTER UNIVERSE FROM TANGENTIAL MODIFIED GRAVITY

As a third example, now we discuss a model proposed in [11]. Such model is defined by means of the function

$$f(R) = R - F(R) = R - a\left[\tanh\left(\frac{b\,(R-R_0)}{2}\right) + \tanh\left(\frac{bR_0}{2}\right)\right] , \tag{21}$$

where a, b and R_0 are arbitrary parameters. One immediately sees that $F(0) = 0$, as required and, moreover, that

$$\lim_{R\to\infty} F(R) = 2\Lambda_{\rm eff} \equiv a\left[1 + \tanh\left(\frac{bR_0}{2}\right)\right] . \tag{22}$$

If $R \gg R_0$ in the present universe then $\Lambda_{\rm eff}$ plays the role of the effective cosmological constant. We also observe that the derivative

$$f'(R) = 1 - \frac{ab}{2\cosh^2\left(\frac{b(R-R_0)}{2}\right)} \tag{23}$$

has a minimum when $R = R_0$, which reads

$$f'(R_0) = 1 - \frac{ab}{2}\,. \tag{24}$$

In order to avoid antigravity one needs to require

$$0 < f'(R) < f'(R_0) = 1 - \frac{ab}{2}\,. \tag{25}$$

The model given by Eq. (21) is able to describe late acceleration, since, in general, de Sitter critical points exist. They are the solutions of (4). Then, we set

$$K(R) = 2f(R) - Rf'(R)\,, \tag{26}$$

and study (numerically) the zeros of this transcendental function (see Fig. 5). We see that for suitable choices of the parameters there are one or two de Sitter critical points.

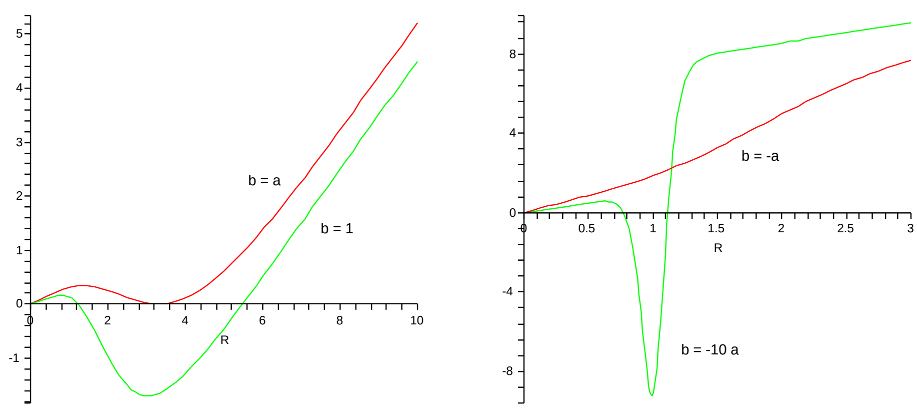

FIG. 5: $K(R)$ for $R_0 = 1, a = 1.64$ and different choices of the parameter b: $b = 1$, $b = a$ (picture on the left); $b = -a$, $b = -10a$ (picture on the right).

Now, we impose R_0 to be a de Sitter critical point for the model in Eq. (21). This means that (26) has to be satisfied for $R = R_0$ and such a condition fixes one parameter,

say a. Then, we get

$$a = \frac{2R_0}{bR_0 - 4\tanh(bR_0/2)}\,, \tag{27}$$

and introducing, for convenience, the dimensionless variables

$$x = \frac{R}{R_0}\,, \qquad\qquad b_0 = R_0 b\,, \tag{28}$$

we obtain

$$f'(R_0) = 1 - \frac{b_0}{b_0 - 4\tanh(b_0/2)}\,. \qquad\qquad \Lambda_{\rm eff} = \frac{1+\tanh(b_0/2)}{b_0 - 4\tanh(b_0/2)}\,, \tag{29}$$

$$K(R) = -\frac{(1-x)\tanh(b_0/2) + 4\tanh(b_0(x-1)/2) + b_0 x \tanh^2(b_0(x-1)/2)}{b_0 - 4\tanh(b_0 x/2)}\,. \tag{30}$$

In order to have $\Lambda_{\rm eff} > 0$ and, at the same time, to avoid antigravity, the parameter b_0 has to be negative and in the range $b_0 < 4\tanh(b_0/2) \sim -3.83$. By varying b_0 in that range, the value of $\Lambda_{\rm eff}$ can acquire any desired value. Moreover the model has always two de Sitter critical points with constant curvatures R_1 and R_2. One can see that $R_2 = R_0$, while $R_0/2 < R_1 < R_0$. In Fig. (6) we have plotted $K(R)$ for some values of b_0. Thus, our

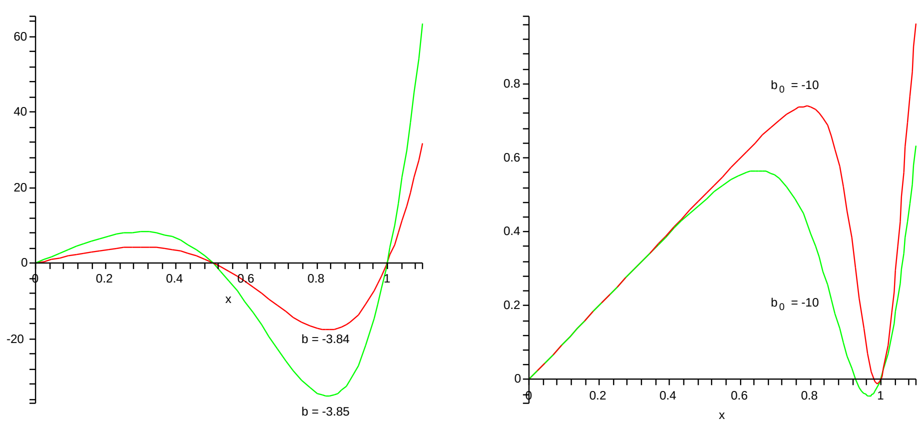

FIG. 6: $K(R)$, with R_0 as a fixed de Sitter solution, for different choices of the parameter b_0: $b_0 = -3.84$, $b_0 = -3.85$ (picture on the left); $b_0 = -10$, $b_0 = -20$ (picture on the right).

study shows the existence of multiply-de Sitter universes in more complicated, tangential models. One can prove that other viable $F(R)$ gravities introduced in ref. [11] naturally lead to de Sitter solutions too, as demonstrated in Ref. [15].

V. BLACK HOLE SOLUTIONS AND RELATED THERMODYNAMICAL QUANTITIES

In this section we will discuss spherically symmetric exact solutions for the modified gravities above. We mainly concentrate on multiply Black Hole solutions and on their related entropies and free energies.

We have seen in the introduction that, requiring constant curvature solutions, one may consider the simplified form of the equations of motion given in (5). As a result, $f(R)$ modified gravity models admit the following class of general static neutral black hole solutions—in four dimensions and with a non vanishing cosmological constant. In order to describe them, recall the metric

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Sigma_k^2\,, \qquad V(r) > 0\,, \tag{31}$$

where $k = 0, \pm 1$ and the possible horizon manifolds are $\Sigma_1 = S^2$, the two dimensional sphere, $\Sigma_0 = T^2$, the two dimensional torus, and $\Sigma_{-1} = H^2/\Gamma$, the two dimensional compact Riemann surface. The scalar curvature for the *ansatz* (31) reads

$$R = -\frac{1}{r^2}[r^2 V'' + 4rV' + 2V - 2k] \tag{32}$$

and this means that every constant curvature solution with $R = R_0$ has to satisfy the equation

$$r^2 V'' + 4rV' + 2V - 2k = -r^2 R_0\,. \tag{33}$$

The general solution of this differential equation depends on two integration constants, b and c, and reads

$$V(r) = \frac{b}{r^2} + k - \frac{c}{r} - \frac{R_0}{12}\,r^2\,. \tag{34}$$

The Ricci tensor and the scalar curvature have to satisfy the equations

$$R_{\mu\nu} = \frac{R_0}{4}\,g_{\mu\nu}\,, \quad R = R_0\,, \tag{35}$$

where R_0 is implicitly given by (4), that is

$$R_0 = \frac{2f(R_0)}{f'(R_0)}\,, \tag{36}$$

Now, it is easy to verify that Eqs. (35) are satisfied only if $b = 0$, while c is an arbitrary parameter, which is usually assumed to be non-negative and related to the mass M of the black hole by $c = 2MG$. The special case $c = 0$ is also admissible. Then, we have

$$V(r) = k - \frac{c}{r} - \frac{R_0}{12} r^2 \,, \quad c \geq 0 \,, \quad V(r) > 0 \,. \tag{37}$$

In principle there are physical solutions of the latter equation for $k = 0$ and $k = -1$, which give rise, respectively, to a torus topology and a hyperbolic topology for the horizon manifold (the so called topological black holes [16]). Here we are mainly interested in the usual spherical symmetric horizons and so we only consider in detail the case $k = 1$.

As it is well known, in the special case $k = 1, c = 0$, $V(r)$ in (37) is always positive when $R_0 < 0$, and this corresponds to the Anti de Sitter (AdS) solution. On the contrary, when $R_0 > 0$, $V(r)$ is positive, for $r < 2\sqrt{3/R_0}$, and this corresponds to the de Sitter solution. If $k = 1$ and $c = 2MG > 0$, one has black hole solutions but only if

$$c^2 R_0 - \frac{16}{9} \leq 0 \implies \alpha \equiv \frac{3}{2} MG\sqrt{R_0} \leq 1 \,. \tag{38}$$

We see that (38) is always satisfied if $R_0 < 0$. This corresponds to the Schwarzschild-Anti-de Sitter (SAdS) black hole. In this case $r > r_H$, r_H being the positive root of $V(r) = 0$ (horizon radius). If $R_0 > 0$, there are solutions only if $\alpha < 1$. In this case $r_H \leq r \leq r_C$, r_H and r_C being the positive roots of $V(r) = 0$ (horizon and cosmological radius respectively). The extremal case $\alpha = 1$ is also admissible (Nariai solution), but the thermodynamics of such a black hole have to be discussed separately [17].

At this point, we provide a brief discussion regarding the thermodynamical properties of the above black hole solutions. If one make use of the Noether charge method for evaluating the entropy associated with the black hole solutions with constant curvature [18] in modified $f(R)$ gravity models, one has [19]

$$S = \frac{A_H}{4G} f'(R_H) \,. \tag{39}$$

where the factor $f(R)$ has to be evaluated on the horizon, with area A_H. As a consequence, one obtains a modification of the “Area Law”. Several examples have been discussed in [19]. In the case of a constant curvature solution one has simply $R_H = R_0$, R_0 being the solution of Eq. (4), which has been investigated in previous Sections. With regard to

this, we have found sufficient conditions to have two de Sitter solutions. Thus, we may investigate their thermodynamical behavior evaluating the associated free energy.

The free energy $\mathcal{F}$ is related to the canonical partition function Z by

$$\mathcal{F} = -\frac{\log Z}{\beta}\,. \tag{40}$$

On the other hand, a semiclassical approximation gives

$$Z \simeq e^{-I_E}\,, \tag{41}$$

where I_E is the Euclidean classical action associated with the de Sitter solution. A direct calculation leads to

$$I_E = -\frac{24\pi f(R_0)}{GR_0^2} \tag{42}$$

and from (4) and (39) it directly follows that

$$I_E = -S_H \Longrightarrow \mathcal{F} = -\frac{S_H}{\beta_H}\,, \tag{43}$$

S_H and β_H being, respectively, the entropy and the inverse temperature of the black hole.

For a generic Schwarzschild-de Sitter (SdS) solution, Eq. (43) reads

$$\mathcal{F} = -\frac{2\pi T_H r_H^2 f(R_0)}{GR_0}\,. \tag{44}$$

The temperature T_H is related to the horizon radius r_H by

$$T_H = \frac{1}{\beta_H} = \frac{|V'(r_H)|}{4\pi} = \frac{1}{4\pi}\left|\frac{2MG}{r_H^2} - \frac{r_H R_0}{6}\right| \tag{45}$$

r_H being a positive solution of the algebraic equation

$$r_H^3 - \frac{12}{R_0}r_H + \frac{24MG}{R_0} = 0\,. \tag{46}$$

As a result, we finally have

$$\mathcal{F} = -\left|r_H - 3MG\right| \frac{f(R_0)}{GR_0}\,. \tag{47}$$

In the pure de Sitter case $M=0$, $r_H = 2\sqrt{3/R_0}$, and so one has

$$\mathcal{F}_{dS} = -2\sqrt{3}\,\frac{f(R_0)}{GR_0^{3/2}}\,. \tag{48}$$

For SdS one needs to consider separately the two admissible cases $\alpha = 1$ and $\alpha < 1$. Here we discuss the second one only, that is, the proper SdS solution where $\alpha = (3/2)MG\sqrt{R_0} < 1$. In this case, Eq. (46) has one negative root and two distinct positive roots, r_H, the event horizon and r_C, the cosmological horizon, with $r_H < r_C$. The positive roots can be written in the form

$$r_C = \frac{4\gamma_C}{\sqrt{R_0}}\,, \qquad \frac{1}{2} < \gamma_C < 1\,, \tag{49}$$

$$r_H = \frac{4\gamma_H}{\sqrt{R_0}}\,, \qquad 0 < \gamma_H < \frac{1}{2}\,, \tag{50}$$

where

$$\gamma_C = \frac{1}{2}\left[\frac{1}{\left(\sqrt{\alpha^2-1}-\alpha\right)^{1/3}} + \left(\sqrt{\alpha^2-1}-\alpha\right)^{1/3}\right] \tag{51}$$

and

$$\gamma_H = \frac{1}{2}\left(\sqrt{3}\sqrt{1-\gamma_C^2}-\gamma_C\right)\,. \tag{52}$$

In this way, we get

$$\mathcal{F}_C = \frac{|2\gamma_C - \alpha|}{\sqrt{3}}\,\mathcal{F}_{dS}\,, \tag{53}$$

$$\mathcal{F}_H = \frac{|2\gamma_H - \alpha|}{\sqrt{3}}\,\mathcal{F}_{dS}\,, \tag{54}$$

$\mathcal{F}_{dS}$ being the expression (48). It has to be noted that, in the latter equations, the factors in front of $\mathcal{F}_{dS}$ are always smaller than 1 and thus the energies $\mathcal{F}_C$ and $\mathcal{F}_H$ are always smaller than $\mathcal{F}_{dS}$, independently of the value of the mass. Moreover, the factor $|2\gamma_C - \alpha|/\sqrt{3}$ is a monotone decreasing function of α, which is equal to 1 for $\alpha = 0$ and to 0 for $\alpha = 1$, while the factor $|2\gamma_H - \alpha|/\sqrt{3}$ is exactly equal to 0 for $\alpha = 0, 1$ and quite small otherwise. Its maximum value, a little bit more than 1/10, is reached for $\alpha \sim 0.8$. This means that the minimum value for the free energy $\mathcal{F}_H$ of a SdS black hole is obtained for $M \sim 0.53/(G\sqrt{R_0})$.

Now, we shall study some models and explicitly compute the corresponding free energy. First of all, as a trivial example, we consider the ΛCDM model described by $f(R) = R - 2\Lambda$. This has a stable de Sitter solution, with $R_0 = 4\Lambda$. Then, we immediately have

$$\mathcal{F}_{dS} = -\frac{\sqrt{3}}{2G\sqrt{\Lambda}}. \tag{55}$$

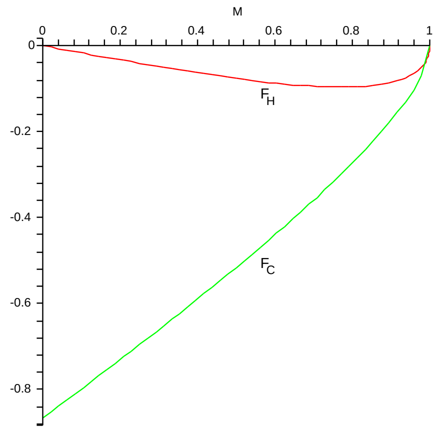

FIG. 7: $\mathcal{F}_C$ and $\mathcal{F}_H$ SdS free energies for the ΛCDM model.

The corresponding SdS cosmological and black-hole free energies are plotted in Fig. 7, as a function of the mass M (in energy units $G\sqrt{\Lambda}$).

As a second example we consider the viable model in (11), with $k = 4$, $c_1 = 1.12 \cdot 10^{-3}$, $c_2 = 6.21 \cdot 10^{-5}$, $c_3 = 0$, as discussed in Sect. II. With this choice of parameters, Eq. (4) has three real solutions for $R_0 = 0$ (Minkowski), $R_1 \simeq 15.33\mu^2$ and $R_2 \simeq 34.92\mu^2$. The last one (R_2) is stable, while the other two are unstable. For the free energies of the two de Sitter solutions we get

$$\mathcal{F}_{dS}^{(1)} = -2\sqrt{3}\,\frac{f(R_1)}{R_1^{3/2}} \simeq -\frac{0.078}{\mu G}\,, \qquad \mathcal{F}_{dS}^{(2)} = -2\sqrt{3}\,\frac{f(R_2)}{R_2^{3/2}} \simeq -\frac{0.286}{\mu G}\,. \tag{56}$$

We see that

$$\mathcal{F}_{dS}^{(2)} < \mathcal{F}_{dS}^{(1)}\,, \qquad \frac{\mathcal{F}_{dS}^{(2)}}{\mathcal{F}_{dS}^{(1)}} \sim 3.636\,. \tag{57}$$

independently of the parameter μ and so the stable de Sitter solution with curvature R_2 is favorite, from the energetic point of view, with respect to the one with smaller curvature R_1.

The SdS free energies are plotted in Fig. 8, as functions of the mass M (in energy units $1/\mu G$), for $0 \leq M < 2/(3G\sqrt{R_1})$. However, it has to be stressed that, for the SdS corresponding to the de Sitter solution with curvature R_2, the mass needs to be restricted to $0 < M < 2/(3G\sqrt{R_2}) < 2/(3G\sqrt{R_1})$ and thus, for a fixed mass M, we can compare the corresponding free energies in this range only. For both the cosmological and black-hole horizons there are critical values of the black hole mass M, say $M_C < M_H$, for which

$\mathcal{F}^{(1)}_{C,H} = \mathcal{F}^{(2)}_{C,H}$. Then, we can distinguish three different regions:
(a) $M < M_C$: in this first case $\mathcal{F}^{(2)}_{C,H} < \mathcal{F}^{(1)}_{C,H}$;
(b) $M_C < M < M_H$: in this second case $\mathcal{F}^{(2)}_{C} < \mathcal{F}^{(1)}_{C}$, but $\mathcal{F}^{(2)}_{H} > \mathcal{F}^{(1)}_{H}$;
(c) finally, for $2/(3G\sqrt{R_1}) > M > M_H$, we have $\mathcal{F}^{(2)}_{C,H} > \mathcal{F}^{(1)}_{C,H}$.

As we have seen from general considerations, the free energies corresponding to the cosmological horizons are monotonous functions of the mass, while the ones corresponding to the black-hole horizons are convex functions, which reach the minimum at $M \sim 0.53/(G\sqrt{R_0})$. For our two cases, R_1 and R_2, we get

$$M_1 \sim \frac{0.53}{G\sqrt{R_1}} \sim \frac{0.135}{\mu G}\,, \qquad \mathcal{F}^{(1)}_{H,min} \sim -\frac{0.008}{\mu G}\,, \tag{58}$$

$$M_2 \sim \frac{0.53}{G\sqrt{R_2}} \sim \frac{0.089}{\mu G}\,, \qquad \mathcal{F}^{(2)}_{H,min} \sim -\frac{0.032}{\mu G}\,. \tag{59}$$

Then we see that, for the model in (11) we are dealing with, the configuration with the minimum free energy corresponds to SdS with mass $M = M_2 \sim 0.089/(\mu G)$ and curvature $R = R_2 \sim /34.92\mu^2$. The associated free energy is given in (59). We also observe that this is smaller than the free energy of the pure de Sitter configuration as it is clear from (56)

Thus, it is demonstrated here that multiple de Sitter solutions can appear also under the form of SdS solutions. In other words, the number of multiply solutions becomes significantly bigger. With the appearance of both dS and SdS universe solutions one can suggest various scenarios for the universe evolution. For instance, we can conjecture that a (pre-)inflationary universe is described by some SdS spacetime. As time proceeds, this universe decays and enters into the well known radiation/matter dominance phase. In its further evolution, the universe transits to the dS (or almost dS) era, by stability and least-energy principle considerations. The future universe may again appear as an SdS spacetime.

VI. DISCUSSION

.

In summary, we have investigated in this paper several viable models of modified gravity which satisfy both the constraints of local as well as cosmological tests. By means

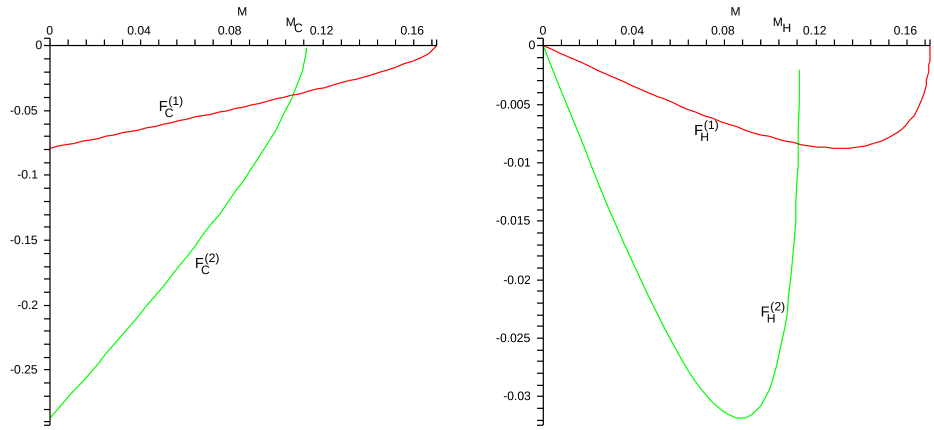

FIG. 8: SdS free energies for the two de Sitter critical points of the model (11): cosmological free energies on the left and black-hole free energies on the right.

of a numerical study, it is demonstrated that some versions of highly non-linear models exhibit multiply de Sitter universe solutions which often appear in pairs, being one of them stable and the other unstable. The numerical evolution of the effective equation of state parameter is presented too. As a result, these models can be considered as natural candidates for the unification of early-time inflation with late-time acceleration through dS critical points. Moreover, based on the de Sitter solutions, multiply SdS solutions can also be constructed. Further, we have investigated the thermodynamic properties of these SdS universes: their corresponding entropies and free energies have been calculated and compared. SdS universe might also appear at the (pre-)inflationary stage.

Owing to the highly non-linear structure of the theories under discussion, the dS universes had to be constructed with numerical tools mainly. Moreover, in order to simplify the problem, at this first stage of the investigation we did not to consider matter contributions. It is clear that, in the next step, we must necessarily include matter and reconsider the problem in its presence. This has the potential to lead to a sufficiently realistic quantitative description of the universe expansion history, in which modified gravity would be responsible for both acceleration stages: the inflation epoch and the dark energy one. This quantitative analysis will be presented elsewhere.

Acknowledgements. This paper is an outcome of the collaboration program INFN (Italy) and DGICYT (Spain). It has been also supported in part by MEC (Spain), projects

FIS2006-02842 and PIE2007-50I023, by AGAUR (Generalitat de Catalunya), contract 2005SGR-00790 and grant 2008BE1-00180, and by RFBR, grant 06-01-00609 (Russia). The work of P.T. was partially supported by RFBR, grant 08-02-00923, and with the scientific school grant 4899.2008.2 of the Russian Ministry of Science and Technology. PT thanks Alexey Toporensky for some useful discussions. The research of EE is partly based on work done while on leave at the Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755, USA.

---

[1] S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007) [arXiv:hep-th/0601213].

[2] S. Nojiri and S.D. Odintsov, arXiv:0807.0685[hep-th]; T. Sotiriou and V. Faraoni, arXiv:0805.1726[gr-qc]; S. Capozziello and M. Francaviglia, Gen.Relat.Grav. **40**, 357 (2008); F. Lobo, arXiv:0807.1640[gr-qc].

[3] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003) [arXiv:hep-th/0307288].

[4] S. Capozziello, Int.J.Mod.Phys. D **11**, 483 (2002); M. Ruggiero and L. Iorio, arXiv:gr-qc/0607093; A. Cruz-Dombriz and A. Dobado, arXiv:gr-qc/0607118; N. Poplawski, arXiv:gr-qc/0610133; M. C. B. Abdalla et al, Class. Quant. Grav. **22**, L35 (2005) [arXiv:hep-th/0409177]; A. Brookfield, C. van de Bruck and L. Hall, arXiv:hep-th/0608015; H. Oyaizu, M. Lima and W. Hu, arXiv:0807.2462[astro-ph]; B. Li, K. Chan and M. Chu, arXiv:astro-ph/0610794; X. Jin, D. Liu and X. Li, arXiv:astro-ph/0610854; T. Sotiriou and S. Liberati, arXiv:gr-qc/0604006; T. Sotiriou, arXiv:0805.1160[gr-qc]; I. Navarro and K. Van Acoleyen, arXiv:gr-qc/0611127; A. Bustelo and D. Barraco, arXiv:gr-qc/0611149; F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B **646**, 105 (2007) [arXiv:hep-th/0612220]; B. Li and J. Barrow, arXiv:gr-qc/0701111; V. Faraoni, arXiv:gr-qc/0703044; arXiv:0706.1223[gr-qc]; arXiv:0805.1249[gr-qc]; arXiv:0810.2602[gr-qc]; O. Bertolami, C. Boehmer, T. Harko and F. Lobo, arXiv:0704.1733[gr-qc]; O. Bertolami and J. Paramos, arXiv:0805.1241[gr-qc]; S. Carloni, A. Troisi and P. Dunsby, arXiv:0706.0452[gr-qc]; arXiv:0707.0106[gr-qc]; arXiv:0809.3673[astro-ph]; J. Santos, J. Alcaniz, F. Carvalho et al, arXiv:0708.0411[astro-ph]; arXiv:0807.2443[astro-ph];

H. Mohseni Sadjadi, arXiv:0709.2435[gr-qc]; N. Lanahan-Tremblay and V. Faraoni, arXiv:0709.4414[gr-qc]; N. Goheer, J. Leach and P. Dunsby, arXiv:0710.0814[gr-qc]; arXiv:0809.5247[gr-qc]; O. Lecian and G. Montani, arXiv:0807.4428[gr-qc]; T. Harko, arXiv:0810.0742[gr-qc]; B. Vakili,arXiv:0809.4591[gr-qc]; S. Srivastava, arXiv:0809.1950[gr-qc]; D. Saez,arXiv:0809.1311[hep-th]; K. Bamba et al, arXiv:0803.3384[hep-th], Phys.Rev. D**77**, 123532 (2008); arXiv:0801.0954[astro-ph], JCAP **0804**,024 (2008); arXiv:0807.2575[hep-th]. P. Mahato, Annales de la Fondation Louis de Broglie 32 (2007) 297-310;arXiv:0710.3821 [astro-ph].

[5] S. Nojiri and S. Odintsov, Gen. Rel. Grav. **36**, 1765 (2004) [arXiv:hep-th/0308176]; Phys. Lett. B **576**, 5 (2003) [arXiv:hep-th/0307071]; Phys. Lett. **B599**, 137 (2004) [arXiv:astro-ph/0403622]; arXiv:0801.4843[astro-ph]; G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. **D73**, 084007 (2006), [arXiv:hep-th/0601008]; S. Capozziello et al, arXiv:astro-ph/0501426; arXiv:hep-th/0512118; G. Allemandi, A. Borowiec, M. Francaviglia and S. D. Odintsov, arXiv:gr-qc/0504057; G. Allemandi, M. Francaviglia, M. Ruggiero and A. Tartaglia, arXiv:gr-qc/0506123; T. Multamaki and I. Vilja, arXiv:astro-ph/0612775; arXiv:0709.3422[astro-ph]; J. A. R. Cembranos, Phys. Rev. D **73**, 064029 (2006) [arXiv:gr-qc/0507039]; T. Koivisto and H. Kurki-Suonio, arXiv:astro-ph/0509422; T. Clifton and J. Barrow, arXiv:gr-qc/0509059; I. Brevik, arXiv:gr-qc/0601100; T. Faulkner, M. Tegmark, E. Bunn and Y. Mao, arXiv:astro-ph/0612569; S. Capozziello et al, arXiv:0804.2163[gr-qc]; arXiv:0712.2980[gr-qc]; arXiv:0810.2549[gr-qc]; S. Nojiri, S. D. Odintsov and P. Tretyakov, arXiv:0704.2520[hep-th]; M. Movahed, S. Baghram and S. Rahvar, arXiv:0705.0889[astro-ph]; C. Boehmer, T. Harko and F. Lobo, arXiv:0710.0966[gr-qc]; J. Miritzis, arXiv:0708.1396[gr-qc]; D. Vollick, arXiv:0710.1859[gr-qc]; J. Evans, L. Hall and P. Caillol, arXiv:0711.3695[astro-ph]; T. Clifton, arXiv:0807.4682[gr-qc]; H. Stefancic, arXiv:0807.4428[gr-qc]; A. Dev et al, arXiv:0807.3445[hep-th]; T. Kobayashi and K. Maeda, arXiv:0807.2503[astro-ph];

[6] S. Nojiri, S. D. Odintsov, Phys. Rev. D **74**, (2006) 086005 [arXiv:hep-th/0608008]; arXiv:hep-th/0611071; S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B **639**, 135 (2006) [arXiv:astro-ph/0604431]. S. Tsujikawa, K. Uddin and R. Tavakol,

arXiv:0712.0082; S. Fay, S. Nesseris and L. Perivolaropoulos, arXiv:gr-qc/0703006. G. Cognola, M. Gastaldi and S. Zerbini, arXiv:gr-qc/0701138; J.L. Cortes and J. Indurain, arXiv:0805.3481[astro-ph]; S. Tsujikawa and T. Tatekawa, arXiv:0804.4343[astro-ph]; N. Deruelle, M. Sasaki and Y. Sendouda, Phys.Rev. D **77**,124024 (2008); K. Kainulainen and D. Sunhede, arXiv:0803.0867[gr-qc]; G. Cognola and S. Zerbini, J.Phys. A **39**, 6245 (2006).

[7] W. Hu and I. Sawicki, arXiv:0705.1158[astro-ph].

[8] S. A. Appleby and R. A. Battye, arXiv:0705.3199[astro-ph]; L. Pogosian and A. Silvestri, arXiv:0709.0296[astro-ph]; S. Tsujikawa, arXiv:0709.1391[astro-ph]; S. Capozziello and S. Tsujikawa, arXiv:0712.2268[gr-qc].

[9] S. Nojiri and S. D. Odintsov, arXiv:0706.1378[hep-th]; arXiv:0707.1941[hep-th].

[10] S. Nojiri and S. D. Odintsov, arXiv:0710.1738[hep-th], Phys.Rev.D **77**,026007 (2008); arXiv:0804.3519[hep-th], Phys.Rev. D **78**, 046006 (2008);

[11] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009 (2008) [arXiv:0712.4017 [hep-th]].

[12] Shin'ichi Nojiri, Sergei D. Odintsov, Petr V. Tretyakov, arXiv:0710.5232[hep-th].

[13] S. Nojiri and S.D. Odintsov, Phys. Lett. B **649**, (2007) 440, hep-th/0702031.

[14] G. Cognola and S. Zerbini, "Homogeneous cosmologies in generalized modified gravity," arXiv:0802.3967 [hep-th].

[15] S. Capozziello, M. De Laurentis, S. Nojiri and S.D. Odintsov, arXiv:0808.1335[hep-th];

[16] L. Vanzo, Phys. Rev. D **56**, 6475 (1997) [arXiv:gr-qc/9705004].

[17] R. Bousso and S. W. Hawking, Phys. Rev. D **57**, 2436 (1998) [arXiv:hep-th/9709224].

[18] R. M. Wald, Phys. Rev. D **48**, R3427 (1993) [arXiv:gr-qc/9307038].

[19] I.Brevik, S. Nojiri, S.D. Odintsov and L. Vanzo, hep-th 0401073, Phys.Rev. D **70**, 043520 (2004); G. Cognola, S. Nojiri, S.D. Odintsov and S. Zerbini, hep-th/0501096, JCAP **0502**, 010 (2005); F. Briscese and E. Elizalde, Phys. Rev. D **77**, 044009 (2008) [arXiv:0708.0432 [hep-th]].